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Optimal boundary control of one-dimensional multi-span vibrating systems

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Abstract

A class of optimal control problems of one dimensional coupled vibrating systems with control applied at the coupled points is considered. A maximum principle is developed for a class of such optimal problems governed by N linear hyperbolic partial differential equations of second order in time and fourth order in space with variable coefficients. The maximum principle given involves a Hamiltonian which contains an adjoint variable as well as an admissible boundary control function. The proof of the maximum principle is given with the help of convexity arguments. The uniqueness theory of the solution of the optimal control problem is given using convexity arguments. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

In recent years one of the very active area of mathematical control theory has been the investigation of the boundary control of linear elastic systems (see e.g. [10] for a comprehensive survey of the area).

The present problem is to study the boundary control of serially connected one dimensional structures. This study is motivated by problems in structural dynamics such as cables modelled by string equations, bridges and flexural members in large space structures modelled by beam equations. These structures are dynamic in nature and require good design of controls. Motivated by these considerations, a novel approach is used to control the vibrations of a flexible structure that consists of a large number of components coupled end to end in the form of a chain, such as coupled vibrating strings [1, 2, 11], or beams [3]. These vibrating structures are distributed parameter systems (DPS),

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which are modelled by partial differential equations, as distinguished from lumped parameter systems (LPS), which are described by ordinary differential equations.

The case of lumped parameter systems has reached a high degree of development [15], while that of distributed parameter systems is still in progress [4, 17]. However the numerical analysis of related algorithms and computational experience is lagging behind theory. This study considers a class of distributed parameter systems consisting of N serially connected structures with control effected through intermediate boundary conditions. In particular, optimal boundary control of a damped multi-span structure is studied with the objective of minimizing a given performance index in a given period of time. The performance index is taken as a quadratic function of the state variable and its derivatives and can be used to represent the potential energy and kinetic energy of the system. A functional of the control is attached to this functional as a penalty term. A maximum principle is developed for a class of damped distributed parameter systems described by N serially connected structures with control effected through interior boundary conditions. In the development, adjoint variables are introduced along with a Hamiltonian which depends on the adjoint variables. The adjoint variables are related to the state variables through certain terminal conditions. The maximum principle gives a relationship, via the Hamiltonian, between the optimal boundary control and the adjoint variable corresponding to the optimal control. The theory presented in this study is capable of providing a method of solution for boundary control of one-dimensional multi-span structures.

Boundary stabilization of elastic systems is presently the subject of extensive work. In [8], a comparative study is made of five methods for calculating the optimal control function for a linear parabolic tracking problem with boundary control. The main emphasis of this paper is placed on obtaining solutions via the Riccati equations and using the open loop solutions for confirmation. Boundary control of a linear differential equation that describes the temperature distribution and displacement within a one-dimensional thermoelastic rod is examined in [6]. In particular, it is shown that temperature or heat flux control at an end point is sufficient to obtain exact null-controllability.

Controllability and stability of a system of coupled strings of nonconstant wave speeds with control applied at the coupled points are studied in [7]. It is shown that the system is approximately controllable if and only if related systems of uncoupled strings do not share a common eigenvalue. The stability problem in the case of constant wave speeds has been considered by Chen et al. [2], Liu [11], and Liu et al. [12]. There is some similarity between their results and the results obtained by Ho [7]. For a similar system representing a chain of coupled vibrating strings, it was shown by Chan et al. [1] that the associated semigroup satisfies the assumption of spectrum determined growth, and hence obtain conditions for energy to decay strongly or exponentially. Chen et al. [3] considered a type of structures that can be formed by N serially connected Euler–Bernoulli beams, with N actuators and sensors co-located at node points. It was shown that when these N beams are strongly connected at all intermediate nodes and their material coefficients satisfy certain properties, uniform exponential stabilization can be achieved by stabilizing at one end point of the composite beam. More general controllability results are obtained in [9], in a sense they apply to planar networks of vibrating beams consisting of several Timoshenko beams connected to each other by rigid joints at all interior nodes of the system.

The particular focus of this study is to seek conditions on the interior boundary control of serially connected damped flexible structures. The optimality condition is derived in the form of a maximum principle. To the best of our knowledge to date very little research has been done on the problem of

interior boundary control of multiple-link structures from the point of view of distributed parameter systems. Closest to the spirit of the present study is the work of Sadek et al. [16] dealing with optimal distributed control of a damped two-span beam, and that of Melvin [14], where the case of more general serially connected structures with distributed control is implemented through a finite number of actuators at a discrete number of points within the structure.

2. Optimal control problem formulation

2.1. System description

Consider a large structure of length L which is made up of N serially connected simple structural elements as shown in Fig. 1, where each structural element s , $s \in \mathbb{N} = \{i \mid 1 \leq i \leq N\}$, is represented by a line segment $\bar{\Omega}_s = [x_{s-1}, x_s]$, where $x_{s-1} < x_s$. The spatial coordinate variable and the time variable will be denoted by $x \in \Omega = \bigcup_{s \in \mathbb{N}} \bar{\Omega}_s$, and $t \in T = (0, t_f) \subset \mathcal{R}^1$, respectively, where t_f is a given positive real number (terminal time). In this study, the endpoints of the component structures will be referred to as *nodes*; the nodes where two or more structural elements meet will be called *multiple*, while the nodes corresponding to only one structural element will be called *simple*.

We shall assume in this study that the simple nodes are fixed, while the multiple nodes will be allowed to oscillate freely or subject to control. We now introduce the following notations:

$$I_S = \{x_0, x_N\}, \quad x_0 = 0, \quad x_N = L,$$

$$I_M = \{x_1, x_2, \dots, x_{N-1}\}, \quad D_s = \Omega_s \times T,$$

$$\Omega_s = (x_{s-1}, x_s),$$

$$\Gamma_s = I_s \times \bar{T}, \quad \Gamma_M = I_M \times \bar{T},$$

$$\mathbb{N}^- = \{1, 2, \dots, N-1\}, \quad \mathbb{N}_0 = \{0, N\}.$$

Let the motion of the one-dimensional structure be governed by the system of partial differential equations

$$\rho_s[w] = m_s(x)\partial_t^2 w(x, t) + d_s(x)\partial_t w(x, t) - L_s[w] = f(x, t), \quad (x, t) \in D_s, \quad (2.1)$$

where $s \in \mathbb{N}$. The system relates the displacement $w(x, t)$ of each s th structure to the applied force distribution $f(x, t)$ at each point $(x, t) \in \bar{D}_s$. $m_s(x)$ is the mass distribution per unit length of the

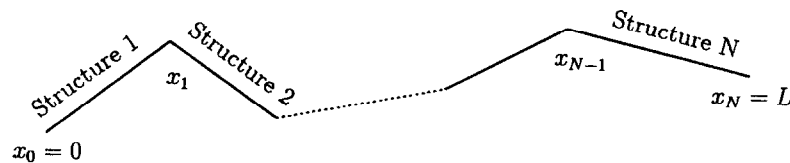


Fig. 1. Serially connected structures.

sth structure and $d_s(x)$ its damping coefficient. The operators L_s are linear time-invariant partial differential operators given by

$$L_s[\cdot] = \sum_{i=0}^2 (-1)^{i+1} \partial_x^{2-i} [P_{2-i}^s(x) \partial_x^{2-i} \cdot], \quad x \in \delta_s, \quad s \in \mathbb{N}, \quad (2.2)$$

where $P_j^s(x)$, $j=0,1,2$ are real-valued functions defined on Ω_s . The notation ∂_z denotes differentiation with respect to z .

Eq. (2.1) are subject to geometric and/or natural boundary conditions of the form

$$B_s^j[w] = \sum_{i=0}^1 \delta_{ij} \partial_x^i [w(x, t)] + \sum_{i=2}^3 \delta_{ij} \partial_x^{i-2} [P_2^s(x) \partial_x^2 w(x, t)] = 0 \quad \text{on } I_S, \quad (2.3)$$

where, $s \in \mathbb{N}_0$, $j=0,1,2$, or 3 and δ_{ij} is the Kronecker delta. Note that two j values are needed for each $s \in \mathbb{N}_0$ to express the geometric boundary conditions of the 4th order differential operators L_s defined by Eq. (2.2). The terms in Eq. (2.3) represent the deflections, slopes, bending moments and shear forces of the structural elements which are zero at the simple nodes $x \in I_S$. The intermediate general boundary conditions at x_s , $s \in \mathbb{N}^-$ are given as follows:

$$C_s^k[w] = \sum_{i=0}^1 \delta_{ik} \partial_x^i [w(x_s^-, t) - w(x_s^+, t)] = 0 \quad \text{on } I_M \quad (2.4)$$

for $k=0,1$ and

$$\begin{aligned} E_s^j[w] &= \delta_{1j} [P_1^s(x_s^-) \partial_x w(x_s^-, t) - P_{-1}^{s+1}(x_s^+) \partial_x w(x_s^+, t)] \\ &\quad + \sum_{i=2}^3 \delta_{ij} \partial_x^{i-2} [P_2^s(x_s^-) \partial_x^2 w(x_s^-, t) - P_2^{s+1}(x_s^+) \partial_x^2 w(x_s^+, t)] \\ &= \delta_{lj} u_s(t) \quad \text{on } I_M \end{aligned} \quad (2.5)$$

for $j=1,2,3$ and some $l \in \{1,2,3\}$.

In Eq. (2.5), u_s are interior boundary controls applied at the coupled points x_s , $s \in \mathbb{N}^-$. The conditions in (2.4) state that the deflections and slopes must be equal at the multiple nodes I_M . While the conditions (2.5) state that slopes, bending moments, and shear forces at the common support (multiple nodes) must be equal to control applied. The conditions (2.4)–(2.5) are usually named as *continuity conditions*.

For given functions $w^0(x)$, $w^1(x) \in L^2(0, L)$, we assume initial conditions

$$w(x, 0) = w^0(x), \quad \partial_t w(x, 0) = w^1(x), \quad x \in \bar{\Omega} \quad (2.6)$$

to hold a.e. on $[0, L]$. Eqs. (2.1)–(2.6) characterize the motion of a large vibrating structure which is formed by N serially connected components coupled end to end in form of a chain with $(N-1)$ controllers applied at the multiple nodes. This formulation includes one-dimensional coupled vibrating strings and beams with controls applied at the coupled points.

2.2. Optimal control problem

In this study we are concerned with suppression of vibrations in the model described in Section 2.1 by the control $t \mapsto u(t) = [u_1(t), u_2(t), \dots, u_{N-1}(t)]^T$ applied at the multiple nodes I_M . The main objective of the control for the structure described by (2.1)–(2.6) is to minimize the index of performance given by

$$J[u] = \sum_{s \in \mathbb{N}} \int_{\Omega_s} \{g_{1s}[x; w(x, t_f), w_x(x, t_f), w_{xx}(x, t_f)] + g_{2s}[x; w_i(x, t_f)]\} dx \\ + \sum_{s \in \mathbb{N}^-} \int_T g_{0s}[t; u_s(t)] dt \quad (2.7)$$

where $w(x, t)$ is the solution of (2.1)–(2.6) and w_x denotes differentiation with respect to x . The properties of the functions g_{0s} , g_{1s} and g_{2s} will be specified in the next section. In applications, the first two terms may represent quantities related to the energy of the physical system. The third term may represent the interior boundary control effort to be used in the control process.

We now consider the definitions of the state space and the space of controls.

Definition 1. A function $(x, t) \in \bar{D}_s \mapsto w(x, t)$ is said to be admissible if w belongs to

$$W_{ad} = \{w \mid w \in L^2(\bar{D}_s), s \in \mathbb{N}\}.$$

Definition 2. A control $t \in \bar{T} \mapsto u(t)$ is said to be admissible if it belongs to the admissible set given by

$$U_{ad} = \{u \mid u_i \in L^2(T), i \in \mathbb{N}^-\}.$$

The boundary optimal control problem to be studied in the present work is the following.

Problem (P). Determine an admissible control u such that

$$J[u^0] \leq J[u], \quad \text{for all } u \in U_{ad}, \quad (2.8)$$

where $J[u]$ is given by (2.7). Such a function u^0 is called an optimal control.

We now make the following assumptions:

(A1) The functions m_s, d_s and P_f^s are defined, bounded and Lebesgue measurable on $\bar{\Omega}_s$ for each $s \in \mathbb{N}$.

(A2) There exists an optimal control for Problem (P).

2.3. Examples

Our interest in the present problem was motivated by the works of Ho [7] and Chen et al. [3], where they dealt with the stabilization of serial strings and beam configurations respectively. The following two motivating examples of second and fourth order models are considered, respectively, subject to general initial conditions (2.6).

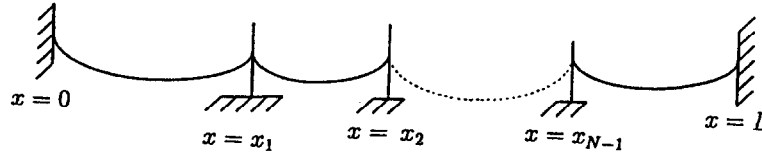


Fig. 2. Serially connected strings.

Example 1. Consider the system of coupled strings with controls applied at the coupled points as shown in Fig. 2. The state equations can be derived from Eq. (2.1) by setting $P_2^s(x)=0$, $P_1^s(x)=T_s$, $P_0^s(x)=0$, $d_s(x)=0$, and $f=0$ for $s \in \mathbb{N}$. The vertical displacements $w(x,t)$, $(x,t) \in \bar{D}_s$ of N connected strings are described by the equations:

$$m_s \partial_t^2 w(x,t) - T_s \partial_x^2 w(x,t) = 0, \quad (x,t) \in D_s, \quad s \in \mathbb{N} \quad (2.9)$$

where T_s and m_s are assumed to be positive constants and can be viewed as the tension and mass density of string s , respectively.

The boundary conditions at the left and right ends $x=0$ and $x=L$ are assumed to be homogeneous, i.e.,

$$w(0,t) = 0 \quad (2.10)$$

or

$$\partial_x w(0,t) = 0, \quad (2.11)$$

$$w(L,t) = 0 \quad (2.12)$$

or

$$\partial_x w(L,t) = 0. \quad (2.13)$$

At the multiple nodes I_M , the strings are coupled. Two coupling transmission conditions are prescribed. They are determined by the equilibrium of forces at the coupling points. The two coupling conditions are of the form

$$w(x_s^-, t) = w(x_s^+, t), \quad s \in \mathbb{N}^-, \quad (2.14)$$

$$T_s \partial_x w(x_s^-, t) - T_{s+1} \partial_x w(x_s^+, t) = u_s(t), \quad s \in \mathbb{N}^-, \quad (2.15)$$

where the conditions (2.14) and (2.15) represent the continuity of the displacement and the discontinuity of the vertical force component, respectively. The functions u_s , $s \in \mathbb{N}^-$ are the actions due to control devices at the multiple nodes I_M . The cost functional J in terms of the system energy E and the control effort F at $t=t_f$ takes in this example the form

$$J[u] = E[u] + F[u], \quad (2.16)$$

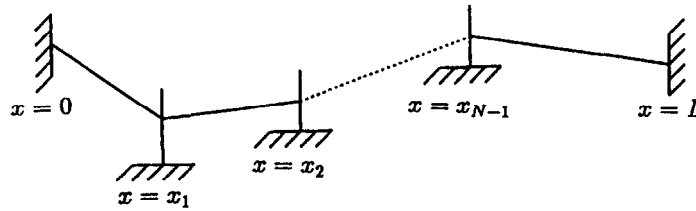


Fig. 3. Serially connected beams.

where

$$E[u] = \frac{1}{2} \sum_{i \in \mathbb{N}} \int_{x_{i-1}}^{x_i} (m_i w_i^2 + T_i w_x^2) dx,$$

$$F[u] = \sum_{i \in \mathbb{N}^-} \int_0^{t_f} \mu_i u_i^2(t) dt.$$

Note that in the cost functional (2.7) we have taken

$$g_{1i}[x; w, w_x, w_{xx}] = T_i w_x^2, \quad i \in \mathbb{N},$$

$$g_{2i}[x; w_i] = m_i w_i^2, \quad i \in \mathbb{N},$$

$$g_{0i}[t; u_i] = \mu_i u_i^2(t), \quad i \in \mathbb{N}^-,$$

where $\mu_i > 0$, $i \in \mathbb{N}^-$, are weighting factors to the control effort F .

Example 2. Consider serially connected beams as shown in Fig. 3, where beam i , $i \in \mathbb{N}$, are represented by line segments $\bar{\Omega}_s = [x_{s-1}, x_s]$, $x_{s-1} < x_s$. We assume that each beam is uniform, with constant density m_s and flexural rigidity $E_s I_s$, $s \in \mathbb{N}$.

The transient vibration of the s th beam is governed by

$$m_s \partial_t^2 w(x, t) + E_s I_s \partial_x^4 w(x, t) = 0, \quad x \in \Omega_s, \quad t \in \bar{T}, \quad s \in \mathbb{N} \quad (2.17)$$

subject to the boundary conditions (simply supported beam)

$$w(0, t) = 0, \quad \partial_x^2 w(0, t) = 0, \quad (2.18)$$

$$w(L, t) = 0, \quad \partial_x^2 w(L, t) = 0. \quad (2.19)$$

The continuity conditions to be satisfied at the nodes $x_s \in I_M$ are

$$w(x_s^-, t) = w(x_s^+, t), \quad (2.20)$$

$$\partial_x w(x_s^-, t) = \partial_x w(x_s^+, t), \quad (2.21)$$

$$E_s I_s \partial_x^3 w(x_s^-, t) = E_{s+1} I_{s+1} \partial_x^3 w(x_s^+, t), \quad (2.22)$$

$$-[E_s I_s \partial_x^2 w(x_s^-, t) - E_{s+1} I_{s+1} \partial_x^2 w(x_s^+, t)] = u_s(t). \quad (2.23)$$

The continuity conditions (2.20)–(2.22) state that the deflections, slopes, and shear forces at the multiple nodes I_M must be equal to each other. The other continuity condition (2.23) is the natural condition arising from the presence of point bending moments $u_s(t)$, $s \in \mathbb{N}^-$ at each point $x_s \in I_M$. The present example can be derived from (2.1)–(2.5) by setting $P_2^s(x) = -E_s I_s$, $P_1^s(x) = 0$, $P_0^s(x) = 0$, $m_s(x) = m_s$, $d_s(x) = 0$, and $f = 0$, $s \in \mathbb{N}$.

In this example we have considered only one combination of boundary conditions. For other possible combinations of boundary conditions, see, e.g., Ref. [13].

In this example we have

$$J[u] = E[u] + F[u], \quad (2.24)$$

where

$$E[u] = \frac{1}{2} \sum_{i \in \mathbb{N}} \int_{x_{i-1}}^{x_i} (m_i w_t^2 + E_i I_i w_{xx}^2) |_{t_i} dx,$$

$$F[u] = \sum_{i \in \mathbb{N}^-} \int_0^{t_i} \mu_i u_i^2(t) dt$$

with $\mu_i > 0$, $i \in \mathbb{N}^-$ being the weighting factors.

Note that in the performance index (2.7), we are taking

$$g_{1i}[x; w, w_x, w_{xx}] = E_i I_i w_{xx}^2, \quad i \in \mathbb{N}$$

$$g_{2i}[x; w_t] = m_i w_t^2, \quad i \in \mathbb{N}^-$$

$$g_{0i}[t; u_i] = \mu_i u_i^2(t), \quad i \in \mathbb{N}^-.$$

3. The maximum principle

3.1. The adjoint problem

Denote $H(\mathbf{R}) = L^2(\mathbf{R})$ the real Hilbert space consisting of all square-integrable real-valued functions on the spatial domain $\mathbf{R} \subset \mathcal{R}^1$ in the Lebesgue sense. The inner product on $H(\mathbf{R})$ is given by

$$\langle f, g \rangle_{H(\mathbf{R})} = \int_{\mathbf{R}} f(x)g(x) dx$$

and its associated norm by

$$\|f\|_{H(\mathbf{R})}^2 = \langle f, f \rangle_{H(\mathbf{R})}$$

for all $f, g \in H(\mathbf{R})$. Similarly, if $D = \mathbf{R} \times T \subseteq \mathcal{R}^2$, the corresponding inner product on $H(D)$ is defined by

$$\langle f, g \rangle_{H(D)} = \int_D f(x, t)g(x, t) dt.$$

In the development of the maximum principle, it is necessary to introduce the adjoint operators \mathfrak{Q}_s^* of \mathfrak{Q}_s , which are defined formally by the relation

$$\langle v, \mathfrak{Q}_s[w] \rangle_{D_s} = \langle \mathfrak{Q}_s^*[v], w \rangle_{D_s}, \quad s \in \mathbb{N} \quad (3.1)$$

for any two functions $w, v \in W_{\text{ad}}$, where the adjoint variable $v \in H(D_s)$ is a solution of the following boundary value problem:

Problem (P*). Determine v satisfying

$$\mathfrak{Q}_s^*[v] = m_s(x) \partial_t^2 v(x, t) - d_s(x) \partial_t v(x, t) - L_s[v] = 0 \quad (3.2)$$

in which the spatial differential operators L_s are defined by (2.2) and with homogeneous boundary conditions at $x \in I_s$

$$B_s^k[v] = 0, \quad k = 0, 1, 2, 3, \quad s \in \mathbb{N} \quad (3.3)$$

and for $x \in I_M$, v satisfies the continuity conditions

$$C_s^k[v] = 0, \quad k = 0, 1, \quad s \in \mathbb{N}^-, \quad (3.4)$$

$$E_s^j[v] = 0, \quad j = 1, 2, 3, \quad s \in \mathbb{N}^-, \quad (3.5)$$

where the spatial differential operators B_s^k , C_s^k and E_s^j are defined by (2.3)–(2.5), respectively.

3.2. Preliminary results

In this section, we prove some lemmas and state a definition that will be of crucial importance in the proof of the maximum principle and the uniqueness of the optimal control. These lemmas are direct generalizations of some results proven in [14].

Let w and $w^0 \in W_{\text{ad}}$ be the state functions of the system (2.1)–(2.6) corresponding to the controls u and u^0 , respectively. Here u^0 is the optimal control and w^0 is the corresponding optimal state. Also, let

$$\Delta w = w - w^0, \quad (3.6)$$

$$\Delta u_s = u_s - u_s^0, \quad s \in \mathbb{N}^-. \quad (3.7)$$

Then Δw satisfies the following equation:

$$\mathfrak{Q}_s[\Delta w] = \Delta f, \quad (3.8)$$

where the operators \mathfrak{Q}_s are defined by (2.1) and Δw satisfies the boundary and continuity conditions

$$B_s^k[\Delta w] = 0, \quad s \in \mathbb{N}, \quad k = 0, 1, 2, 3, \quad (3.9)$$

$$C_s^k[\Delta w] = 0, \quad s \in \mathbb{N}^-, \quad k = 0, 1, \quad (3.10)$$

$$E_s^j[\Delta w] = \delta_{lj} \Delta u_s, \quad s \in \mathbb{N}^-, \quad j = 1, 2, 3, \quad (3.11)$$

where $l \in \{1, 2, 3\}$, the spatial partial differential operators B_s^k , C_s^k and E_s^j are defined by (2.3)–(2.5), respectively, and the initial conditions

$$\Delta w(x, 0) = 0, \quad \Delta w_t(x, 0) = 0. \quad (3.12)$$

Next we let

$$\Delta g_{1s} = g_{1s}[x; w, w_x, w_{xx}] - g_{1s}[x; w^0, w_x^0, w_{xx}^0] \quad \text{at } t = t_f, \quad (3.13)$$

$$\Delta g_{2s} = g_{2s}[x; w_t] - g_{2s}[x; w_t^0] \quad \text{at } t = t_f, \quad (3.14)$$

$$\Delta g_{0s} = g_{0s}[t; u_s(t)] - g_{0s}[t; u_s^0(t)]. \quad (3.15)$$

Definition 3. A real valued functional $(x, \theta) \in \bar{\Omega} \times V \mapsto g[x; \theta]$ is said to be convex in θ , where V is a vector space, if

$$g[x; \lambda\theta_1 + (1 - \lambda)\theta_2] \leq \lambda g[x; \theta_1] + (1 - \lambda)g[x; \theta_2]$$

for all $\theta_1, \theta_2 \in V$ and all $\lambda \in (0, 1)$. If strict inequality holds whenever $\theta_1 \neq \theta_2$, then g is said to be strictly convex.

In what follows we prove three lemmas to be used in the proof of the maximum principle.

Lemma 1. Let Δw be the solution of the state system (3.6)–(3.12) with $l = 1$ in (3.11) and v be the solution of the adjoint system (3.2)–(3.5). If we set

$$I_1[v, \Delta w] := \sum_{s \in \mathbb{N}} \int_{\Omega_s} \{v L_s[\Delta w] - \Delta w L_s[v]\} dx \quad (3.16)$$

where the differential operators L_s are defined by (2.2), then

$$I_1[v, \Delta w] = \sum_{s \in \mathbb{N}^-} v(x_s, t) \Delta u_s(t) \quad (3.17)$$

u_s being the controls applied at the coupled points x_s , $s \in \mathbb{N}^-$.

Proof. Using integration by parts twice in $I_1[v, \Delta w]$ we obtain

$$\begin{aligned} I_1[v, \Delta w] = & \sum_{s \in \mathbb{N}} \{ -v \partial_x [P_2^s(x) \partial_x^2 \Delta w] + \partial_x(v) P_2^s(x) \partial_x^2 \Delta w \\ & + v P_1^s(x) \partial_x \Delta w + \Delta w \partial_x [P_2^s(x) \partial_x^2 v] - \partial_x(\Delta w) P_2^s(x) \partial_x^2 v - \Delta w P_1^s(x) \partial_x v \} |_{\partial \Omega_s}. \end{aligned} \quad (3.18)$$

Applying the continuity conditions (3.4), (3.5), (3.10), and (3.11), (3.18), becomes

$$\begin{aligned} I_1[v, \Delta w] = & \{ -v \partial_x [P_2^s(x) \partial_x^2 \Delta w] + \partial_x(v) P_2^s(x) \partial_x^2 \Delta w + v P_1^s(x) \partial_x \Delta w \\ & + \Delta w \partial_x [P_2^s(x) \partial_x^2 v] - \partial_x(\Delta w) P_2^s(x) \partial_x^2 v - \Delta w P_1^s(x) \partial_x v \} |_{x_0}^{x_N} \\ & + \sum_{s \in \mathbb{N}^-} v(x_s, t) \Delta u_s(t). \end{aligned} \quad (3.19)$$

Using the boundary conditions (3.3) and (3.9) in (3.19) completes the proof. \square

Remark 1. If we change l in (3.11), then by using the same argument as given in (3.19), the expression (3.16) becomes

$$I_1[v, \Delta w] = \sum_{s \in \mathbb{N}^-} \partial_x v(x_s, t) \Delta u_s(t), \quad \text{for } l = 2 \quad (3.20)$$

$$I_1[v, \Delta w] = - \sum_{s \in \mathbb{N}^-} v(x_s, t) \Delta u_s(t), \quad \text{for } l = 3. \quad (3.21)$$

Lemma 2. Let $g_s[x; w, w_x, w_{xx}]|_{t=t_f} \in C^2[\bar{\Omega}_s \times \mathcal{R}^3]$ be a convex function in w, w_x , and w_{xx} for each $s \in \mathbb{N}$. Suppose the following condition is satisfied:

$$\sum_{s \in \mathbb{N}} \{[-\partial_{w_s} g_s + \partial_x(\partial_{w_{xx}} g_s)] \Delta w - (\partial_{w_{xx}} g_s) \Delta w_x\} = 0 \quad \text{on } \partial \Omega_s. \quad (3.22)$$

Then for $g = (g_1, \dots, g_N)$ one has

$$\begin{aligned} I_2[g, \Delta w] &:= \sum_{s \in \mathbb{N}^-} \int_{\Omega_s} [\partial_w g_s - \partial_x(\partial_{w_{xx}} g_s) + \partial_x^2(\partial_{w_{xx}} g_s)] \Delta w \, dx \\ &= \sum_{s \in \mathbb{N}^-} \int_{\Omega_s} [\Delta g_s - R_{1s}(x; \bar{w}, \bar{w}_x, \bar{w}_{xx})] \, dx, \quad \text{at } t = t_f \end{aligned} \quad (3.23)$$

with $\int_{\Omega_s} R_{1s} \, dx \geq 0$, Δg_s defined by

$$\Delta g_s = g_s[x; w, w_x, w_{xx}] - g_s[x; w^0, w_x^0, w_{xx}^0]$$

and $(x; \bar{w}, \bar{w}_x, \bar{w}_{xx})$ is an intermediate point of the line joining $(x; w, w_x, w_{xx})$ to $(x; w^0, w_x^0, w_{xx}^0)$ for each $x \in \Omega_s$ and $s \in \mathbb{N}$.

Proof. Using integration by parts in $I_2[g, \Delta w]$ one gets

$$\begin{aligned} I_2[g, \Delta w] &= \sum_{s \in \mathbb{N}} \left\{ -(\partial_{w_s} g_s) \Delta w + \partial_x(\partial_{w_{xx}} g_s) \Delta w - (\partial_{w_{xx}} g_s) \Delta w_x \right\} \Big|_{\partial \Omega_s} \\ &\quad + \sum_{s \in \mathbb{N}} \int_{\Omega_s} [(\partial_w g_s) \Delta w + (\partial_{w_x} g_s) \Delta w_x + (\partial_{w_{xx}} g_s) \Delta w_{xx}] \, dx. \end{aligned} \quad (3.24)$$

The first term of Eq. (3.24) vanishes in view of the condition (3.22). Applying Taylor's formula on g_s , it can then be given by

$$g_s = g_s^0 + (\partial_w g_s^0) \Delta w + (\partial_{w_x} g_s^0) \Delta w_x + (\partial_{w_{xx}} g_s^0) \Delta w_{xx} + R_{1s}(x; \bar{w}, \bar{w}_x, \bar{w}_{xx}), \quad (3.25)$$

where $g_s^0 = g_s[x; w^0, w_x^0, w_{xx}^0]$, $s \in \mathbb{N}$ and $R_{1s}(x; \bar{w}, \bar{w}_x, \bar{w}_{xx})$ is the Taylor's remainder of order 2 [2]. The convexity of g_s [14] implies that $R_{1s} \geq 0$ and the result (3.23) follows from (3.24)–(3.25). \square

Lemma 3. Let $h_s[x; w_t] \in C^2(\bar{\Omega}_s \times \mathcal{R})$ be a convex function in w_t for each $s \in \mathbb{N}$. Then

$$\begin{aligned} I_3[h, \Delta w_t] &:= \sum_{s \in \mathbb{N}} \int_{\Omega_s} \partial_{w_t} h_s \Delta w_t \, dx \\ &= \sum_{s \in \mathbb{N}} \int_{\Omega_s} [\Delta h_s - R_{2s}(x; \bar{w}_t)] \, dx \end{aligned} \quad (3.26)$$

where $h = (h_1, \dots, h_N)$, $\Delta w_t = w_t - w_t^0$ and $\Delta h_s = h_s[x; w_t] - h_s[x; w_t^0]$ with $\int_{\Omega_s} R_{2s} \, dx \geq 0$ for each $s \in \mathbb{N}$ and \bar{w}_t being an intermediate value between w_t and w_t^0 .

Proof. The result (3.26) follows from Taylor's formula and the convexity of h_s . It is the same kind of proof as given in Lemma 2. \square

3.3. Proof of the maximum principle

In this section we state and prove the maximum principle for Problem (P) for the case $l = 1$ in (2.5). In the proof of the maximum principle we make the following assumptions:

(A3) $g_{1s}[x; w, w_x, w_{xx}]$ is a convex function in w, w_x , and w_{xx} .

(A4) $g_{2s}[x; w_t]$ is a convex function in w_t .

(A5) $g_{0s} \in C^0(D_s \times \mathbb{R}^1)$, $g_{1s} \in C^3(D_s \times \mathbb{R}^3)$, $g_{2s} \in C^2(D_s \times \mathbb{R}^1)$ with g_{1s} satisfying the relation

$$\sum_{s \in \mathbb{N}} \{ [-\partial_{w_x} g_{1s} + \partial_x(\partial_{w_{xx}} g_{1s})] \Delta w - (\partial_{w_{xx}} g_{1s}) \Delta w_x \} |_{\partial \Omega_s} = 0,$$

where Δw denotes a solution of the state system described by (3.8)–(3.12) for a given $\Delta f(x, t)$.

Let us first define the Hamiltonian H of Problem (P) as

$$H[t; v, u] = \sum_{s \in \mathbb{N}^-} \{ v(x_s, t) u_s(t) - g_{0s}[t; u_s] \}, \quad (3.27)$$

where $u_s(t)$ is the control applied at the coupled points $x_s \in I_M$ and v is a solution of the adjoint boundary value Problem (P*) described by (3.2)–(3.5).

Now we state and prove the maximum principle which is a direct generalization of [14]:

Theorem 4. Let $u^0 \in U_{ad}$ be an admissible solution to Problem (P) that satisfies the maximum principle:

$$\max_{\bar{u} \in U_{ad}} H[t; v, u] = H[t; v, u^0] \quad \text{for } t \in T \text{ a.e.} \quad (3.28)$$

where H is given by (3.27) and let v be the solution of the adjoint Problem (P*) given by (3.2)–(3.5). Assume w is a solution of the initial-boundary value Problem (P) described by (2.1)–(2.5) with $l = 1$ and the assumptions (A1)–(A5) are satisfied. Moreover, assume that v and w satisfy the following terminal conditions:

$$\begin{aligned} m_s(x) \partial_t v(x, t_f) - d_s(x) v(x, t_f) &= \{ \partial_w g_{1s}[x; w, w_x, w_{xx}] - \partial_x(\partial_{w_x} g_{1s}[x; w, w_x, w_{xx}]) \\ &\quad + \partial_x^2(\partial_{w_{xx}} g_{1s}[x; w, w_x, w_{xx}]) \} |_{t=t_f} \end{aligned} \quad (3.29)$$

and

$$m_s(x)v(x, t_f) = -\partial_{w_t} g_{2s}[x; w_t]|_{t=t_f} \quad (3.30)$$

for all $s \in \mathbb{N}$. Then u^0 is an optimal control of Problem (P).

Proof. The adjoint operator \mathfrak{L}_s^* of \mathfrak{L}_s satisfies

$$\langle v, \mathfrak{L}_s[\Delta w] \rangle_{D_s} - \langle \mathfrak{L}_s^*[v], \Delta w \rangle_{D_s} = 0, \quad (3.31)$$

that is to say

$$\sum_{s \in \mathbb{N}} \int_T \int_{\Omega_s} \{v \mathfrak{L}_s[\Delta w] - \Delta w \mathfrak{L}_s^*[v]\} dx dt = 0, \quad (3.32)$$

or equivalently

$$\begin{aligned} & \sum_{s \in \mathbb{N}} \int_T \int_{\Omega_s} \{m_s(x)v \partial_t^2 \Delta w - m_s(x) \Delta w \partial_t^2 v + d_s(x)v \partial_t \Delta w \\ & + d_s(x) \Delta w \partial_t v - v L_s[\Delta w] + \Delta w L_s[v]\} dx dt \\ & = 0. \end{aligned} \quad (3.33)$$

By applying Lemma 1, one obtains

$$\sum_{s \in \mathbb{N}} \int_{\Omega_s} \int_T [v(m_s(x) \partial_t \Delta w + d_s(x) \Delta w) - \Delta w m_s(x) \partial_t v]_t dt dx - \sum_{s \in \mathbb{N}^-} \int_T v(x_s, t) \Delta u_s(t) dt = 0. \quad (3.34)$$

The initial conditions (3.12) yield

$$\sum_{s \in \mathbb{N}} \int_{\Omega_s} \left\{ v[m_s(x) \partial_t \Delta w + d_s(x) \Delta w] - \Delta w m_s(x) \partial_t v \right\} \Big|_{t=t_f} dx - \sum_{s \in \mathbb{N}^-} \int_T v(x_s, t) \Delta u_s(t) dt = 0. \quad (3.35)$$

Rearranging terms in (3.35) one obtains

$$- \sum_{s \in \mathbb{N}^-} \int_T v(x_s, t) \Delta u_s(t) dt = \sum_{s \in \mathbb{N}} \int_{\Omega_s} [\Delta w(m_s(x) \partial_t v - d_s(x)v) - m_s(x)v \partial_t \Delta w] \Big|_{t=t_f} dx. \quad (3.36)$$

From the definition of the performance index J given in (2.7), it follows that

$$\Delta J = \sum_{s \in \mathbb{N}} \int_{\Omega_s} \{\Delta g_{1s} + \Delta g_{2s}\}|_{t=t_f} dx + \sum_{s \in \mathbb{N}^-} \int_T \Delta g_{0s} dt \quad (3.37)$$

where $\Delta J = J[u] - J[u^0]$, Δg_{1s} , Δg_{2s} , and Δg_{0s} are given by (3.13)–(3.15), respectively. By Lemmas 2 and 3 with $g_s = g_{1s}$ and $h_s = g_{2s}$ we have

$$\begin{aligned} \Delta J &= \sum_{s \in \mathbb{N}} \int_{\Omega_s} \{[\partial_w g_{1s} - \partial_x(\partial_{w_t} g_{1s}) + \partial_x^2(\partial_{w_{tx}} g_{1s})] \Delta w + (\partial_{w_t} g_{2s}) \partial_t \Delta w\} \Big|_{t=t_f} dx \\ &+ \sum_{s \in \mathbb{N}^-} \int_T \Delta g_{0s} dt + \sum_{s \in \mathbb{N}} \int_{\Omega_s} (R_{1s} + R_{2s}) dx. \end{aligned} \quad (3.38)$$

In view of the terminal conditions (3.29) and (3.30), the equality (3.38) becomes

$$\begin{aligned} \Delta J = & \sum_{s \in \mathbb{N}} \int_{\Omega_s} \{m_s(x) \partial_t v - d_s(x) v\} \Delta w - m_s(x) v \partial_t \Delta w \Big|_{t=t_f} dx \\ & + \sum_{s \in \mathbb{N}^-} \int_T \Delta g_{0s} dt + \sum_{s \in \mathbb{N}^-} \int_{\Omega_s} (R_{1s} + R_{2s}) dx. \end{aligned} \quad (3.39)$$

Inserting (3.36) into (3.39) and noting that $\int_{\Omega_s} (R_{1s} + R_{2s}) dx \geq 0$, we get

$$\Delta J \geq - \sum_{s \in \mathbb{N}^-} \int_T v(x_s, t) \Delta u_s(t) dt + \sum_{s \in \mathbb{N}^-} \int_T \Delta g_{0s} dt. \quad (3.40)$$

So

$$\Delta J \geq \sum_{s \in \mathbb{N}^-} \int_T \{H[t; v, u_s^0] - H[t; v, u_s]\} dt. \quad (3.41)$$

Since u^0 satisfies the maximum principle (3.28) we get $\Delta J \geq 0$. This completes the proof of the maximum principle when $l = 1$ in (2.5). \square

We now consider the following note:

Remark 2. By similar arguments of Theorem 1 a maximum principle can be proved for $l = 2, 3$ in (3.16). One has to take into account Remark 1. Also, Hamiltonian (3.27) becomes

(i) for $l = 2$

$$H[t; v, u] = \sum_{s \in \mathbb{N}^-} \{\partial_x v(x_s, t) u_s(t) - g_{0s}[t; u_s]\}, \quad (3.42)$$

(ii) for $l = 3$

$$H[t; v, u] = - \sum_{s \in \mathbb{N}^-} \{v(x_s, t) u_s(t) + g_{0s}[t; u_s]\}. \quad (3.43)$$

4. Uniqueness of the optimal control

Under the assumption that the functions g_{0s} are strictly convex, the optimal control u^0 is unique.

Theorem 1. Let g_{1s}, g_{2s} satisfy (A3)–(A4) and g_{0s} be strictly convex in w_t for each $s \in \mathbb{N}$. Then there is at most one optimal control to Problem (P).

Proof. Assume that Problem (P) has two solutions u^1 and u^2 . Let w^1 and w^2 in W_{ad} be the respective responses u^1 and u^2 . Since the differential operators \mathfrak{Q}_s described in (2.1), are linear, then $\frac{1}{2}(w^1 + w^2)$ is the response to $\frac{1}{2}(u^1 + u^2)$. It follows by the convexity of g_{1s}, g_{2s} and g_{0s} that

$$\sum_{s \in \mathbb{N}} g_{1s} \left[x; \frac{1}{2}(h^1 + h^2) \right] \leq \frac{1}{2} \sum_{s \in \mathbb{N}} \{g_{1s}[x; h^1] + g_{1s}[x; h^2]\},$$

$$\sum_{s \in \mathbb{N}} g_{2s} \left[x; \frac{1}{2}(w_t^1 + w_t^2) \right] \leq \frac{1}{2} \sum_{s \in \mathbb{N}} \{g_{2s}[x; w_t^1] + g_{2s}[x; w_t^2]\},$$

$$\sum_{s \in \mathbb{N}^-} g_{0s} \left[x; \frac{1}{2}(u^1 + u^2) \right] < \frac{1}{2} \sum_{s \in \mathbb{N}^-} \{g_{0s}[x; u^1] + g_{0s}[x; u^2]\},$$

where $h^i = (w^i, w_x^i, w_{xx}^i)$, $i = 1, 2$ and hence

$$J\left[\frac{1}{2}(u^1 + u^2)\right] < \frac{1}{2}\{J[u^1] + J[u^2]\}. \quad (4.1)$$

Since u^1 and u^2 are optimal boundary controls, we have

$$J[u^1] = \min_{u \in U_{ad}} J[u] = J[u^2]. \quad (4.2)$$

It follows by (4.1)–(4.2) and $\frac{1}{2}(u^1 + u^2) \in U_{ad}$, that

$$J\left[\frac{1}{2}(u^1 + u^2)\right] = \min_{u \in U_{ad}} J[u] \quad (4.3)$$

which is clearly a contradiction. \square

5. Conclusion

A mathematical theory for a class of optimal boundary control problems for a damped distributed parameter system represented by N serially connected flexible structures coupled through boundary conditions is formulated. This class of distributed parameter systems includes coupled vibrating strings and beams. The objective of the boundary control is to minimize a prescribed performance index of a continuous structure over admissible boundary controllers in a given period of time. The state function is subject to various constraints, in particular, systems of partial differential equations coupled through boundary conditions. The damping of the vibrations of the continuous structure is achieved by boundary controllers applied at the connecting points. Guided by the work of Melvin [14], a maximum principle is proved for a system of N partial differential equations of the second order in time and fourth order in space with variable coefficients. Under additional convexity assumptions on the state variables and their derivatives, the optimal boundary control is shown to be unique. The maximum principle thus obtained provides a solution procedure of solving such types of problems.

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